Classical Oscilators in General Relativity [*]

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Abstract

It is shown that in general relativity some static metrics are able to simulate oscillatory motions. Their form depends on two arbitrary real parameters which determine the specific oscillation modes. The conclusion is that these metrics can be used for new geometric models of closed or even open bags.

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1 Introduction

In general relativity the quark confinement can be treated as a pure geometric effect produced by an anti-de Sitter cavity [1, 2]. This model is based on a (3+1)-dimensional anti-de Sitter static metric [2] which reproduces the usual equations of motion of the nonrelativistic (isotropic) harmonic oscillator (NRHO). On the other hand, it is known that the quantum massive scalar field on anti-de Sitter spacetime has an equidistant discrete energy spectrum [3]. Thus, the classical or quantum particle inside of an anti-de Sitter bag appears as the natural relativistic correspondent of the classical or quantum NRHO.

However, we can ask if there are other metrics that may be used for new geometric bag models. In our opinion, these must produce similar kinetic effects as the anti-de Sitter one, namely closed trajectories inside of the bag. Obviously, it is very difficult to classify all the metrics with this property but if we restrict ourselves to the simple particular case of the oscillatory motions, arising from linear equations, then we could find some new interesting metrics. This is the motive why we would like to study here metrics producing timelike geodesics which (I) are linear in space Cartesian coordinates and their time derivatives, and (II) have the NRHO as nonrelativistic limit. These systems will be called generally relativistic oscillators (RO). Our objective is to find a representative class of (3+1) metrics satisfying these two requirements and to analyze the geodesic motion of these models, in order to point out new possible relativistic effects.

We show that there are two interesting types of metrics with these properties. The first one contains deformed or conformaly transformed anti-de Sitter metrics. These describe spherical closed cavities where the confined particles move as harmonic or rotating oscillators. Another type is that of the deformed or conformaly transformed de Sitter metrics of open bags. In these models the particle remains inside of the bag only for energies less than a given limit. If the energy increases over this limit then the particle escapes to infinity.

We start in Sec.2 with a short review of the properties of the relativistic central motion by taking into account the conservation of the energy and angular momentum. In the next section we introduce a family of RO accomplishing (I) and (II), the metrics of which depend on two real parameters.

The classical motion of these RO is studied in Sec.4 where we obtain the mentioned results.

2 The relativistic central motion

In general relativity the correspondent of the nonrelativistic classical central motion is the geodesic motion on central (i.e., spherically symmetric) static charts where the line element in holonomic Cartesian coordinates, $x^0 = ct$ and $x^i, i = 1, 2, 3$, is invariant under time translations and the space rotations, $R \in SO(3)$, of the space coordinates, $x^i \to R^i_j x^j$. The most general form of the line element is

$$ds^2 = A(r)dt^2 - [B(r)\delta_{ij} + C(r)x^i x^j]dx^i dx^j .\tag{1}$$

where A, B and C are arbitrary differentiable functions of $r = |\mathbf{x}|$.

The geodesic equations of a test particle of mass m can be derived directly in terms of $x^i(t)$ and their time derivatives, $\dot{x}^i$ and $\ddot{x}_i$, by using the Hamiltonian formalism. Moreover, this help us to write down the conservation laws given by the Noether theorem. Thus we obtain the conservation of the energy

$$E = mc\, A \frac{dt}{ds} ,\tag{2}$$

and of the angular momentum

$$L^{ij} = E \frac{B}{A}(x^i \dot{x}^j - x^j \dot{x}^i) .\tag{3}$$

With these ingredients, after a few manipulations we find the equations of the timelike geodesics,

$$\ddot{x}^i + \frac{1}{r}\left(\frac{A}{B}\right)' \frac{L^{ij}}{E}\dot{x}^j + \partial_i W = 0 ,\tag{4}$$

written with the notation $' = \partial_r$. Here

$$W = \frac{1}{2}\frac{A}{B + r^2 C}\left(\frac{m^2 c^2}{E^2}A - 1 - \frac{L^2}{E^2}\frac{AC}{B^2}\right)\tag{5}$$

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is a function of r playing the role of potential. These equations explicitly depends on E and L^{ij} or $L = |\mathbf{L}|$, used here as integration constants. Thus we obtain an alternative form of the well-known geodesics of the relativistic central motion [4, 5] from which one can easily recover their main properties. For example, we see that the trajectories are on space shells orthogonal to $\mathbf{L}$. In other respects, we recognize that the second term of (4) is just the Coriolis-like relativistic contribution [6].

The form of the potential (5) is complicated since it is strongly non-linear in A, B and C. For this reason it is convenient to perform a change of functions considering three new functions of r, denoted by α, β and γ, such that

$$A = c^2 \frac{\alpha}{\beta}, \quad B = \frac{\alpha}{\beta\gamma}, \quad C = \frac{\alpha}{\beta^2\gamma} \frac{\gamma - \beta}{r^2}. \tag{6}$$

Then Eqs.(4) become

$$\ddot{x}^i + \frac{\gamma'}{r} \frac{L^{ij}}{E} \dot{x}^j + \partial_i W = 0, \tag{7}$$

where

$$W = \frac{c^2}{2} \left(\alpha \frac{m^2 c^4}{E^2} - \beta + \gamma(\beta - \gamma) \frac{1}{r^2} \frac{L^2 c^2}{E^2} \right) \tag{8}$$

has a more comprehensible form.

The geodesic equations in spherical coordinates can be obtained directly in terms of functions α, β and γ starting with the line element

$$ds^2 = c^2 \frac{\alpha}{\beta} dt^2 - \frac{\alpha}{\beta^2} dr^2 - \frac{\alpha}{\beta\gamma} r^2 (d\theta^2 + \sin^2\theta d\phi^2). \tag{9}$$

If we consider the conservation of the energy and that of the angular momentum chosen along to the 3-axis, we obtain the geodesic equations on the space shell $\theta = \pi/2$,

$$\ddot{r} + w' = 0, \tag{10}$$

$$\dot{\phi} = \frac{\gamma}{r^2} \frac{Lc^2}{E}, \tag{11}$$

where

$$w = \frac{c^2}{2} \left(\alpha \frac{m^2 c^4}{E^2} - \beta + \frac{\beta\gamma}{r^2} \frac{L^2 c^2}{E^2} \right). \tag{12}$$

Notice that the prime integral $\dot{r}^2 + 2w = 0$ allows us to directly integrate the radial equation (10) as in nonrelativistic mechanics.

Thus we obtain new forms of the geodesic equations of the relativistic central motion that have some technical advantages. One of them is that the potentials W or w are linear in α and β ia a such a manner that the potential energy will have a term proportional with $\alpha - \beta$ in the nonrelativistic limit. In addition, from (7) we see that the Coriolis-like relativistic effect is due only to the first derivative of the function γ. Therefore, this effect is canceled when γ is a constant.

3 The metrics of relativistic oscillators

Let us turn now to the problem of our RO which must satisfy the conditions (I) and (II). We shall show that these can be accomplished if we consider the particular functions

$$\alpha_o = 1 + (1 + \lambda)\hat{\omega}^2 r^2, \quad \beta_o = 1 + \lambda\hat{\omega}^2 r^2, \quad \gamma_o = 1 + \lambda'\hat{\omega}^2 r^2, \qquad (13)$$

where $\hat{\omega} = \omega/c$ and λ and λ' are two arbitrary real parameters. Indeed, if we take $L = L^{12} \neq 0$ and $L^{23} = L^{13} = 0$ in order to keep the trajectory on the shell $x^3 = 0$, then Eqs.(7) take the form

$$\ddot{x}^1 + 2\Omega_c \dot{x}^2 + \Omega_0^2 x^1 = 0,$$
$$\ddot{x}^2 - 2\Omega_c \dot{x}^1 + \Omega_0^2 x^2 = 0, \qquad (14)$$

which is linear in x^i and their time derivatives. The constant coefficients herein are the Cartesian effective frequency squared,

$$\Omega_0^2 = \omega^2 \left[(1 + \lambda)\frac{m^2 c^4}{E^2} - \lambda + \lambda'(\lambda - \lambda')\frac{L^2\omega^2}{E^2} \right], \qquad (15)$$

and the Coriolis-like frequency

$$\Omega_c = \lambda' \frac{L\omega^2}{E}. \qquad (16)$$

On the other hand, we observe that in the nonrelativistic limit, for small values of the nonrelativistic energy $E_{nr} = E - mc^2$ and $c \to \infty$, we have

$$\lim_{c \to \infty} \Omega_0 = \omega, \qquad \lim_{c \to \infty} \Omega_c = 0, \qquad (17)$$

which means that all these RO lead to NRHO. Notice that in the absence of the interaction ($\omega \to 0$) the metric becomes a flat one.

Thus we have found a family of metrics which satisfy (I) and (II). According to (1), (6) and (13), these metrics are given by the line elements in Cartesian coordinates

$$ds^2 = c^2 \frac{\alpha_o}{\beta_o} dt^2 - \frac{\alpha_o}{\beta_o \gamma_o} \left[\delta_{ij} - (\lambda - \lambda') \frac{\hat{\omega}^2}{\beta_o} x^i x^j \right] dx^i dx^j , \qquad (18)$$

while in spherical coordinates the line elements can be obtained directly by replacing (13) in (9).

These metrics represent in some sense a generalization of the anti-de Sitter one. In order to avoid supplementary singularities we restrict ourselves to $\lambda' \geq 0$. Then we see that the metrics with $\lambda < 0$ are either deformations, if $\lambda' \neq 0$, or conformal transformations, if $\lambda' = 0$, of some anti-de Sitter metrics. These are are singular on the sphere of the radius $r_e = c/\omega\sqrt{-\lambda}$ which is just the event horizon of an observer situated at $x^i = 0$. Particularly, $\lambda = -1$ and $\lambda' = 0$ correspond to the anti-de Sitter metric [2]. The metrics with $\lambda > 0$ and $\lambda' \neq 0$ are deformations of some de Sitter static metrics while for $\lambda' = 0$ these are just conformal transformations of the de Sitter one. However, the de Sitter metric is not included in this family since it is not able to produce oscillations.

In general, the metrics we have introduced can be considered as solutions of the Einstein equations with sources but only in the presence of the cosmological term which must give rise to the anti-de Sitter metric when the space is devoid of matter. Our preliminary calculations indicate that, by choosing suitable values of cosmological constant, one can find large domains of parameters where the stress energy tensor of the gravitational sources satisfies at least the dominant energy condition [7, 8].

4   Trajectories and oscillation modes

The values of the parameters λ and λ' determine the oscillation modes of RO. In Cartesian coordinates these appear as two orthogonal harmonic oscillators coupled through velocities because of the Coriolis-like term. One can verify that they have two oscillation modes with the frequencies $\Omega_\pm =$

$\Omega \pm \Omega_c$. Hence it results that our RO are not isotropic harmonic oscillators if $\Omega_c \neq 0$.

However, these oscillation modes can be better analyzed in spherical coordinates. From (10)-(12) and (13) we have

$$\ddot{r} + \Omega^2 r - \frac{L^2 c^4}{E^2} \frac{1}{r^3} = 0, \qquad (19)$$

$$\dot{\phi} = \frac{Lc^2}{E} \frac{1}{r^2} + \Omega_c, \qquad (20)$$

where

$$\Omega^2 = {\Omega_0}^2 + {\Omega_c}^2 = \omega^2 \left[(1+\lambda) \frac{m^2 c^4}{E^2} - \lambda + \lambda\lambda' \frac{L^2 \omega^2}{E^2} \right] \qquad (21)$$

is the radial frequency squared. These equations can be easily integrated obtaining the solutions

$$r(t) = \frac{1}{\sqrt{\Omega}} [\kappa_1 + \kappa_2 \sin 2(\Omega t + \delta_1)]^{1/2}, \qquad (22)$$

$$\phi(t) = \delta_2 + \Omega_c t + \arctan \left\{ \frac{E}{Lc^2} [\kappa_1 \tan(\Omega t + \delta_1) + \kappa_2] \right\}, \qquad (23)$$

where

$$\kappa_1 = \frac{c^2}{2\Omega} \left[1 - \frac{m^2 c^4}{E^2} - (\lambda + \lambda') \frac{L^2 \omega^2}{E^2} \right] \qquad (24)$$

and

$$\kappa_1{}^2 - \kappa_2{}^2 = \frac{L^2 c^4}{E^2}. \qquad (25)$$

The phases δ_1 and δ_2 are the remaining integration constants after we have fixed the values of the energy and of the three components of the angular momentum. These phases are determined by the initial conditions (at $t = 0$)

$$r(0) = r_0, \qquad \phi(0) = \phi_0. \qquad (26)$$

The Eqs.(22) and (23) are similar to those of an isotropic harmonic oscillator of the frequency Ω apart the second term of Eq.(23) which produces an uniform rotation of the angular velocity Ω_c. This means that the RO with $\lambda' \neq 0$ are in fact relativistic *rotating oscillators*. However, when $\lambda' = 0$ we have $\Omega_c = 0$ and, consequently, the RO becomes an harmonic oscillator. If,

in addition, we take $\lambda = -1$ then we obtain the anti-de Sitter oscillator of Ref.[2].

According to (25) the motion is possible only if

$$\kappa_1 \geq \frac{Lc^2}{E} . \tag{27}$$

This condition combined with (24) gives us the energy spectrum for a fixed L as

$$E^2 \geq (mc^2 + L\omega)^2 - (1 + \lambda - \lambda')L^2\omega^2 . \tag{28}$$

On the other hand, we observe that for $\lambda > 0$ there are values of E and L for which the oscillatory motion degenerates into open (uniform or accelerated) motions. This is because the oscillations appear only when

$$\Omega^2 > 0 \tag{29}$$

which, according to (21), gives the oscillation condition

$$\lambda E^2 < (1 + \lambda)m^2c^4 + \lambda\lambda'L^2\omega^2 . \tag{30}$$

When both these conditions are accomplished we have an oscillatory motion with a trajectory of an ellipsoidal form on the ring $r \in [r_{min}, r_{max}]$ where

$$r_{min} = \sqrt{\frac{\kappa_1 - \kappa_2}{\Omega}} , \qquad r_{max} = \sqrt{\frac{\kappa_1 + \kappa_2}{\Omega}} . \tag{31}$$

Particularly, for $L = 0$ we obtain the same result as in the case of (1+1) RO, namely that the particles of the models with $\lambda > 0$ oscillate only if $mc^2 < E < mc^2\sqrt{1 + 1/\lambda}$ while those having $\lambda \leq 0$ oscillate for all the possible energies, $E > mc^2$ [9].

What is interesting here is that for all $\lambda \neq -1$ and $\lambda' \neq 0$ the frequency Ω depends on E and L. This can be interpreted as a pure relativistic effect since in the nonrelativistic limit $\Omega \to \omega$, as it results from (17). Moreover, then we have

$$\kappa_1 \to \frac{E_{nr}}{\omega m}, \qquad \kappa_2 \to \frac{1}{\omega m}\sqrt{E_{nr}^2 - L^2\omega^2} \tag{32}$$

such that the trajectory becomes just the well-known nonrelativistic one. Notice that in the particular case of the harmonic oscillators (with $\lambda' = 0$) the frequencies depend only on E like in the case of the (1+1) RO [9].

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5 Conclusions

Here we have defined a new family of RO as a possible generalization of the anti-de Sitter oscillator. The RO with $\lambda < 0$ are models of closed bags since their metrics are singular on the sphere of the radius r_e and, therefore, the oscillating particle remains confined to cavity interior. We specify that our new parameter λ can be seen as a supplementary fit parameter of the cavity radius. The metrics with $\lambda > 0$ have no singularities and, therefore, the free particles can have either closed trajectories, when the motion is oscillatory because of the values of E and L satisfying the condition (30), or open trajectories if this condition is not fulfilled. Hence these metrics are of new possible models of open bag from which the particle can escape when it has enough energy.

Thus we have an example of a family of metrics giving oscillatory motions with specific kinetic effects. The next step may be to classify all the metrics able to generate oscillations and to establish the physical context in which such metrics can be produced by gravitational sources. In our opinion this problem is sensitive and must be carefully analyzed. We hope that the method presented here should be useful for further developments in this direction. On the other hand, our family of metrics is interesting from the quantum point of view since all the models of quantum spinless RO with $\lambda' = \lambda + 1$ are analytically solvable [10]. Thus we can compare the classical and quantum kinetic effects of a large set of simple models. This could be useful for better understanding the relation between the classical and quantum approaches in general relativity.

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